Quantum Information Theory

Final exam
Spring term 2023

Assignment date: July 7th, 2023, 9h15
Due date: July 7th, 2023, 12h15

PHYS-550 – Exam – room CM 1 120

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.

  Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions**.
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers: Problem 1	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
Total	/100

Short Answer Questions

1. a) Explain what is meant by the reduced state of a multi-qubit system. (2 marks)

   b) Compute the reduced states on system A of the following systems:

   (i) $\frac{1}{\sqrt{2}}(|\psi_-\rangle_{AB} + |00\rangle_{AB})$

   (ii) $\alpha\,|0, 0, \dots, 0\rangle_{AB_1 \dots B_n} - \beta\,|+, +, \dots, +\rangle_{AB_1 \dots B_n}$

   (3 marks)

2. Consider the state

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \tag{1}$$

   where α is real.

   a) What values does α need to take to be a valid quantum state? (4 marks)

   b) State two different ensemble decompositions for this state. (Please provide at least one explicitly, the second may be described geometrically). (3 marks)

   c) Hence (or otherwise) explain what is meant by 'the ensemble ambiguity paradox'? (2 marks)

3. a) Show that

$$||U_A U_B - U_C U_D||_p \leq ||U_A - U_C||_p + ||U_B - U_D||_p \tag{2}$$

   for any unitaries U_A, U_B, U_C, U_D. (4 marks)

   b) Show that Eq. (2) also holds for $p = 1$ for non-unitary matrices U_j as long as $||U_j||_\infty \leq 1$ for $j = A, B, C, D$.

   (2 marks)

4. Consider the matrices:

$$A_0 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad A_1 = \sqrt{1 - |a|^2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3}$$

a) Determine for which values of a these matrices define a valid quantum channel $\mathcal{E}$ on a qubit. (3 marks)

b) Determine for which choices of a the channel $\mathcal{E}$ sends $|0\rangle\langle 0|$ to the maximally mixed state. (3 marks)

5. a) Show that the Von Neumann entropy of a state $\rho = \sum_i p_i \sigma_i \otimes |\lambda_i\rangle\langle\lambda_i|$, where $\{|\lambda_i\rangle\}$ form an orthogonal basis, is given by $S(\rho) = H(\{p_i\}) + \sum_i p_i S(\sigma_i)$. (4 marks)

   b) What is the entropy of the following states:

   (i) $\rho = \frac{1}{2}|+0\rangle\langle +0| + \frac{1}{2}|01\rangle\langle 01|$
   (ii) $\rho = \frac{2}{3}|\psi_-\rangle\langle\psi_-| + \frac{1}{3}|\psi_+\rangle\langle\psi_+|$

   (2 marks)

6. a) Show that $U\sigma_Y = \sigma_Y U^*$ for any single qubit unitary U. (4 marks)

   b) Find the operator χ such that $|\psi_-\rangle = |\mathrm{vec}(\chi)\rangle$. (1 mark)

   c) Hence show that $(U \otimes U)|\psi_-\rangle = |\psi_-\rangle$ (3 marks)

7. Define the resource theory of entanglement. Why is it useful?

   (Please focus on clarity over quantity. A good answer will use a mix of mathematical definitions and conceptual explanation)

   (6 marks)

8. The dephasing channel kills off all coherence in the computational basis. That is,

$$D(\rho) = \langle 0|\rho|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho|1\rangle\, |1\rangle\langle 1|\,. \tag{4}$$

   Using Stinespring's dilation theorem state a quantum circuit to implement the dephasing channel for a single qubit.

   (4 marks)

Long Answer Questions

Please pick 2 questions to answer - mark your choices clearly on the cover sheet.

Question A - Measurement

a) What is a POVM? How does a POVM differ from a measurement of a Hermitian observable? (6 marks)

b) Consider the Chrysler states $|e_k\rangle = \cos(2\pi k/5)|0\rangle + \sin(2\pi k/5)|1\rangle$. Show that $\mathcal{M} = \{\alpha|e_k\rangle\langle e_k|\}_{k=0}^{4}$ form a valid POVM for some choice in α. What value of α is required?
(4 marks)

c) Sketch the Chrysler states on the Block sphere. What are the probabilities for each of the measurement outcomes of $\mathcal{M}$ for the state $\rho = |+_y\rangle\langle +_y|$? (4 marks)

A POVM is informationally complete (IC) if by repeating M many times on a state ρ we can fully reconstruct any state ρ from the measurement outcomes.

Or, equivalently, a POVM is informationally complete on a d-dimensional Hilbert space $\mathcal{H}$ if it consists of *at least* d^2 operators that span $\mathcal{H}$.

d) State an IC-POVM for a single qubit. (1 mark)

e) Does the POVM formulated in terms of the Chrysler states form an informationally complete measurement? If not, how could it be modified such that it would? (4 marks)

A symmetric IC-POVM (SIC-POVM) is one where the (Hilbert-Schmidt) inner product between all elements of the IC-POVM are equal.

SIC-POVMs consisting of *exactly* d^2 elements are called minimal.

f) State a minimal SIC-POVM for a single qubit.

(You may describe/sketch your proposal geometrically rather than provide an explicit expression if that is helpful)

(6 marks)

Question B - Shot Noise

An estimator for performing a measurement $M = \sum_m \lambda_m |\lambda_m\rangle\langle\lambda_m|$ on a state ρ is given by

$$X_N = \frac{1}{N}\sum_{s=1}^{N} \lambda_{m(s)} \tag{5}$$

where $\lambda_{m(s)}$ is the eigenvalue obtained on the s_{th} shot and N is the total number of shots.

a) Show that this estimator is *unbiased*. (3 marks)

b) Show that the variance of the estimator is given by $\text{Var}(X_N) = \frac{1}{N}\text{Var}_\rho(M)$ where $\text{Var}_\rho(M)$ is the standard quantum mechanical variance of an observable M in the state ρ. (6 marks)

The SWAP *test* is shown below

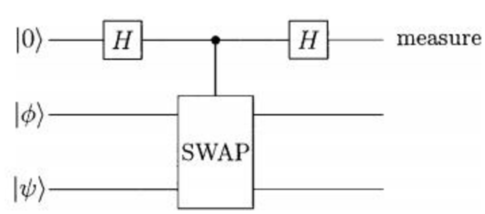

Figure 1: **Swap Test.** Here both $|\psi\rangle$ and $|\phi\rangle$ are arbitrary n-qubit states and the top register is a single ancilla qubit initialised in the $|0\rangle$ state. The Hadamard gate, denoted by H, transforms between the X and Z bases, e.g. $\sigma_X = H\sigma_Z H$. The final measurement is made to just the ancilla qubit in the computational basis.

c) Show that this circuit can be used to calculate $|\langle\psi|\phi\rangle|^2$. (5 marks)

d) State another circuit (using only n-qubits) which could be used to compute the same quantity. (You may assume that you know the circuit to prepare the state $|\psi\rangle$.) (2 marks)

e) Discuss (quantifiably!) how shot noise effects these two circuits as used to calculate $|\langle\psi|\phi\rangle|^2$. (8 marks)

f) Hence discuss the advantages and disadvantages of these two circuits. (1 mark)

Question C - Entanglement Theory

a) What is meant by the entanglement entropy of a bipartite pure state? Why is it a natural measure of entanglement?

For full marks (brief) conceptual explanations as well as mathematical definitions and a (brief) derivation are required.

(8 marks)

b) What deterministic LOCC transformation (one way, two way, no way) is possible between these two states:

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle \text{ and } |\phi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{6}}|11\rangle + \frac{1}{\sqrt{6}}|22\rangle \tag{6}$$

Justify your answer. (6 marks)

c) Why can entanglement entropy not be used as a measure of the entanglement of a mixed state? (2 mark).

d) State the Peres-Horodecki criterion. How does it differ for $n = 2$ qubit states versus $n > 2$ qubit states? (3 marks).

e) Consider the following mixture of two Bell states

$$\rho = p|\psi_-\rangle\langle\psi_-| + (1-p)|\psi_+\rangle\langle\psi_+| \,. \tag{7}$$

Use the Peres-Horodecki criterion to find the range of p for which ρ is entangled. (6 marks).